Lecture 22 (11/15/21)

$[0,1] \to G$

<u>Recall</u>. Two closed curves γ_0, γ_1 in G are homotopic if $\exists$ homotopy $\Gamma : [0,1] \times [0,1] \to G$ s.t. $\Gamma(s,0) = \gamma_0(s)$, $\Gamma(s,1) = \gamma_1(s)$, $\Gamma(0,t) = \Gamma(1,t)$.

We write $\gamma_0 \underset{G}{\sim} \gamma_1$ (or $\gamma_0 \sim \gamma_1$ if G is understood.)

<u>Def.</u> ① If σ is the constant curve $\sigma(s) = a$ for some $a \in G$ and $\gamma \sim \sigma$, then γ is <u>homotopic to zero</u>, $\gamma \sim 0$.

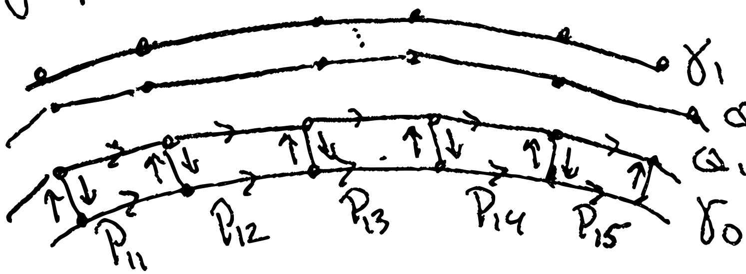
<u>Cauchy's Thm - II</u>. If f is analytic in G and $\gamma \sim 0$ in G, then $\int_\gamma f\,dz = 0$.

<u>Cauchy's Thm - III</u>. If f is anal. in G and $\gamma_0 \sim \gamma_1$ in G, then $\int_{\gamma_1} f\,dz = \int_{\gamma_0} f\,dz$.

Since III $\Rightarrow$ II, we shall only prove III.

A main obstacle in pf is that the closed curves $\gamma_t = \Gamma(\cdot, t)$ need not be p.w. smooth and hence $\int_{\gamma_t}$ may not make sense.

Pf of CT-III. The idea is illustrated by the following picture

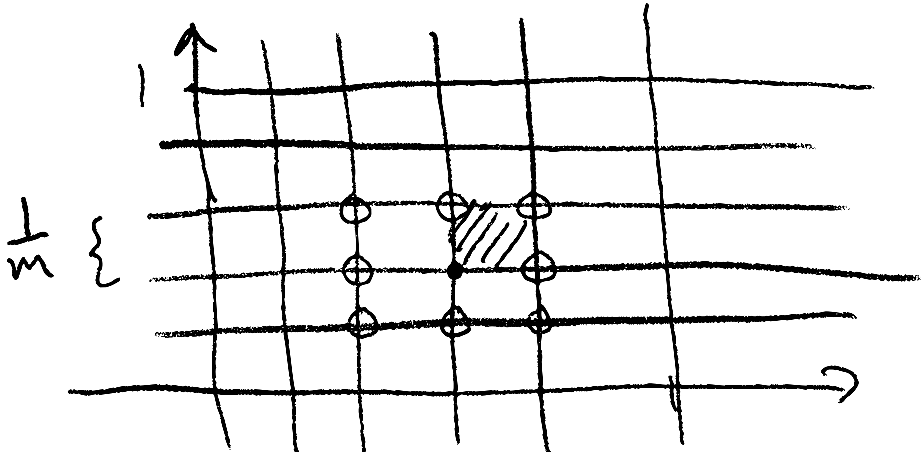


We define closed polygonal paths Q_k, $k=1,\dots,m$, for some large m and show $\int_{\gamma_0} f\,dz = \int_{Q_1} f\,dz = \dots = \int_{Q_m} f\,dz = \int_{\gamma_1} f\,dz$ by using CT-I on polygonal paths P_{jk}. For this, we need to ensure the P_{jk} and interiors are all contained in G.

Thus, let $K = \Gamma([0,1]^2) \subset\subset G$ and $r = d(K, \mathbb{C}\setminus G)$. Since Γ cont., $[0,1]^2$ compact $\Rightarrow$ Γ is unif. cont. and we can partition $[0,1]\times[0,1]$ into a equidistant lattice of $(m+1)^2$ points s.t. the squares satisfy

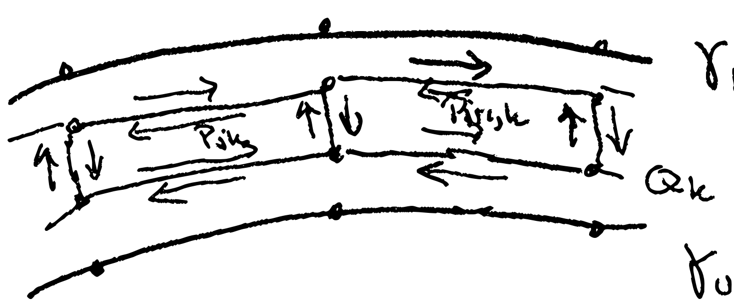
$|\Gamma(s,t) - \Gamma(s',t')| < r/2$ when (∗)

$(s,t), (s',t') \in [\frac{j}{m}, \frac{j+1}{m}] \times [\frac{k}{m}, \frac{k+1}{m}]$.



Let $z_{jk} = \Gamma(\frac{j}{m}, \frac{k}{m})$, $Q_k = \bigcup_{j=1}^{m} [z_{j-1,k}, z_{j,k}]$,

and $P_{jk} = [z_{jk}, z_{j+1,k}] \cup [z_{j+1,k}, z_{j+1,k+1}]$
$\cup [z_{j+1,k+1}, z_{j,k+1}] \cup [z_{j,k+1}, z_{jk}]$



By construction, diam $P_{jk} < r/2$,

$$\Rightarrow \quad P_{jk} \subseteq B(z_{jk}, r/2) \subseteq G.$$

By Baby-CT (CT in $B(a, r)$) which follows from existence of primitive)

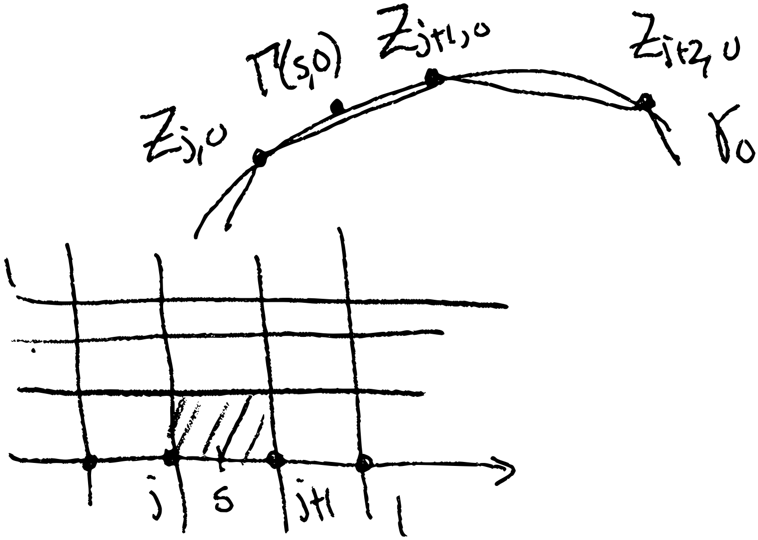
$$\Rightarrow \quad \int_{P_{jk}} f\,dz = 0.$$

For fixed k, when we sum $\int_{P_{jk}}$ over j, the $\downarrow | \uparrow$ integrals over $[z_{j,k}, z_{j,k+1}]$ will cancel out since the Q_k are closed (see pic). The remaining edges add up to Q_k and $-Q_{k+1}$ $\Rightarrow$

$$0 = \sum_{j=1}^{m} \int_{P_{jk}} f\,dz = \int_{Q_k} f\,dz - \int_{Q_{k+1}} f\,dz$$

$$\Rightarrow \int_{Q_{k+1}} f\,dz = \int_{Q_k} f\,dz \Rightarrow \int_{Q_0} f\,dz = \int_{Q_m} f\,dz.$$

But Q_0, Q_{m+1} are polygonal approx. of γ_0 and γ_1. Suffices to show $\int_{\gamma_0} f\,dz = \int_{Q_0} f\,dz$.



by (*)

So let $\sigma_j(s) = \gamma_0(s)$, $s \in \left[\frac{j}{m}, \frac{j+1}{m}\right]$, and hence the closed curve $\sigma_j - [Z_{j,0}, Z_{j+1,0}]$ is $\subseteq B(Z_{j,0}, r/2)$. By CT-Baby,

$$\int_{\sigma_j} f\,dz = \int_{[Z_{j,0}, Z_{j+1,0}]} f\,dz$$

Summing again $\Rightarrow \int_{\gamma_0} f\,dz = \int_{Q_0} f\,dz$

and same argument shows $\int_{\gamma_1} f\,dz = \int_{Q_n} f\,dz$

and the proof is complete. $\boxtimes$

<u>Cor 1.</u> $\gamma \sim 0$ in $G \Rightarrow n(\gamma, z) = 0$, $\forall z$ in $\mathbb{C} \setminus G$.

<u>Rem.</u> If Cor 1 could be shown <u>directly</u> then CT-II would follow from CT-I. This is done in HW, <u>assuming</u> the curves Γ_t are p.w. smooth.

Def. 2 Two curves $\gamma_0, \gamma_1 : [0,1] \to G \subseteq \mathbb{C}$ s.t. $\gamma_0(0) = \gamma_1(0) = a$, $\gamma_0(1) = \gamma_1(1) = b$ are <u>Fixed End Point (FEP) homotopic</u>, $\gamma_0 \sim \gamma_1$, if $\exists$ FEP-homotopy $\Gamma : [0,1]^2 \to G$ s.t. $\Gamma(\cdot, 0) = \gamma_0$, $\Gamma(\cdot, 1) = \gamma_1$ and $\Gamma(0,t) = a$ and $\Gamma(1,t) = b$.

Prop 1. If γ_0, γ_1 from a to b are FEP-homotopic in G, then the closed curve $\gamma_0 - \gamma_1$ is homotopic to 0.

Pf. See construction in Conway or DIY. $\boxtimes$

Independence of Path Thm. Let f be analytic in G, and γ_0, γ_1 FEP-homotopic curves in G. Then

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz.$$

Def 3. A region $G \subseteq \mathbb{C}$ is simply connected if every closed (cont!) curve is homotopic to 0.

Cor. 2. (Cauchy's Thm - IV). Let f be analytic in simply connected region G. Then, $\int_\gamma f\,dz = 0$ $\forall$ closed curve γ in G.

Cor 3. ($\exists$ primitive). Let f be analytic in simply connected region G. Then f has a primitive F analytic in G.

Pf of Cor 3. Fix $a \in G$ and for every $z \in G$ let γ_z be a curve from a to z. Set

$$F(z) = \int_{\gamma_z} f \, d\zeta .$$

The fact that $F' = f$ now follows from Cor 2 as in pf of Morera. $\boxtimes$

Cor 4. Let G be simply connected region and f an analytic fcn in G s.t. $f \neq 0$. Then, $\exists$ a branch of $\log f(z)$ in G, i.e. an analytic function g s.t. $e^{g(z)} = f(z)$. The branch is unique if $e^{w_0} = f(z_0)$ and we require $g(z_0) = w_0$.

Pf of Cor 4. Since f'/f is analytic in G, it has a primitive g by Cor 3. By adding a constant we choose g uniquely s.t. $g(z_0) = w_0$. We compute

$$\frac{d}{dz}(fe^{-g}) = f'e^{-g} - fg'e^{-g} = e^{-g}f\left(\frac{f'}{f} - g'\right)$$

$$= 0.$$

Thus, $fe^{-g} =$ constant and by plugging in $z = z_0$ and using $g(z_0) = w_0$ we see that $fe^{-g} = 1$ or $e^{g} = f$.

Reversing the argument shows g is unique. ☒